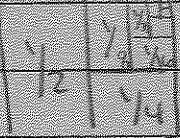
Day 4 - AM

Series

We've already studied sequences

$\{a_n\}_{n=1}^{\infty} = a_1, a_2, a_3, \ldots$

A natural question: what is $\underbrace{\sum_{n=1}^{\infty} a_n}_{\text{infinite series}}$?

Ex: 

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = 1?$

Partial Sum $S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + \ldots + a_N$

defn: $\sum_{n=1}^{\infty} a_n = \lim_{N \to \infty} S_N$

Ex: $a_n = \frac{1}{2^n}$ $\quad \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$

$a_n = \frac{1}{n^2}$ $\quad \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots$

If $\lim_{N \to \infty} S_N = S$, then we say $S = \sum_{n=1}^{\infty} a_n$.

If S doesn't exist, $\sum_{n=1}^{\infty} a_n$ "diverges"

if $S_N \to \infty$, series diverges to ∞.

Ex: Telescoping Series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1(2)} + \frac{1}{2(3)} + \frac{1}{3(4)} + \frac{1}{4(5)} + \ldots$$

Using partial fractions, $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

So $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$

$= (1 - 1/2) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots$

so $S_1 = 1 - 1/2$

$S_2 = (1 - \cancel{1/2}) + (\cancel{1/2} - \frac{1}{3}) = 1 - \frac{1}{3}$

$S_3 = (1 - \cancel{1/2}) + (\cancel{1/2} - \cancel{\frac{1}{3}}) + (\cancel{1/3} - 1/4) + \dots = 1 - 1/4$

thus $S_N = 1 - \frac{1}{N+1}$

So $\sum_{n=1}^{\infty} a_n = \lim_{N \to \infty} 1 - \frac{1}{N+1} = \boxed{1}$.

<u>Thm:</u> $\sum (a_n \pm b_n) = \sum (a_n) \pm \sum (b_n)$

$\sum (c a_n) = c \sum (a_n)$

FYI... its often hard to determine what a series converges to... settle for determining convergence/divergence.

<u>Ex: Geometric Series</u>

$\sum_{n=0}^{\infty} cr^n = c + cr + cr^2 + cr^3 + \dots$

$S_N = c + cr + cr^2 + cr^3 + \dots + cr^N$

$rS_N = cr + cr^2 + cr^3 + \dots + cr^N + cr^{(N+1)}$

$S_N - rS_N = c - cr^{(N+1)} = c(1 - r^{(N+1)})$

$S_N(1-r) = c(1 - r^{(N+1)}) \Rightarrow S_N = \frac{c(1 - r^{N+1})}{(1-r)}$ if $r \neq 1$ $\longrightarrow$

Then $\sum_{n=0}^{\infty} cr^n = \lim_{N\to\infty} S_N$

$$= \lim_{N\to\infty} \frac{c(1-r^{N+1})}{1-r} = \begin{cases} \frac{c}{1-r} & \text{if } |r|<1 \\ \text{diverges} & \text{if } |r|\geq 1 \end{cases}$$

Thm: Sum of Geometric Series

if $c\neq 0$ and $|r|<1$,

$$\sum_{n=0}^{\infty} cr^n = c + cr + cr^2 + \dots = \frac{c}{1-r}$$

$$\sum_{n=M}^{\infty} cr^n = cr^M + cr^{M+1} + cr^{M+2} + \dots = \frac{cr^M}{1-r}$$

Ex: $\sum_{n=0}^{\infty} 2^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \overset{\text{use thm}}{=} \frac{1}{1-1/2} = \frac{1}{1/2} = \boxed{2}$

Ex: $\sum_{n=4}^{\infty} 8\left(\frac{-1}{3}\right)^n \overset{\text{use thm}}{=} \frac{8\left(\frac{-1}{3}\right)^4}{1-(-1/3)} = \frac{8\left(\frac{1}{81}\right)}{4/3} = \frac{\cancel{8}^{\,2}}{\cancel{81}_{\,27}} \cdot \frac{\cancel{3}}{\cancel{4}} = \boxed{\frac{2}{27}}$

Thm: (Divergence Test)

if $a_n \not\to 0$, then $\sum_{n=1}^{\infty} a_n$ diverges

Ex: converge or diverge?

A) $\sum_{n=1}^{\infty} \frac{n}{3n+12}$

$$\lim_{n\to\infty} \frac{n}{3n+12} = \frac{1}{3} \neq 0 \quad \boxed{\text{diverges}}$$

B) $\cos\left(\frac{1}{2}\right) + \cos\left(\frac{1}{3}\right) + \cos\left(\frac{1}{4}\right) + \dots$

$$\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \cos\left(\lim_{n\to\infty} \frac{1}{n}\right) = \cos(0) = 1 \neq 0 \quad \boxed{\text{diverges}}$$

Careful!

Ex: $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \dots$

$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ but,

$S_N = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{N}} \geq \frac{1}{\sqrt{N}} + \frac{1}{\sqrt{N}} + \dots \frac{1}{\sqrt{N}} = \frac{N}{\sqrt{n}} = \sqrt{N}$

then $\lim_{N\to\infty} S_N = \lim_{N\to\infty} \sqrt{N} = \infty$

So $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges

Thm: $\sum_{n=1}^{\infty} \frac{1}{n^p} = \sum_{n=1}^{\infty} n^{-p}$ converges if $p>1$ and diverges otherwise

There are alot more theorems on Convergence! Check your textbook!